Module 2

Simplex Method

Setting up simplex method

original form:

Ex: max $Z = 3x_1 + 5x_2$

s.t $x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

change inequality to equality (i.e Add a new variable to the equation ~~of the form~~ $\leq$)

$$\left.\begin{aligned} x_1 + x_3 &= 4 \\ 2x_2 + x_4 &= 12 \\ 3x_1 + 2x_2 + x_5 &= 18 \\ Z - 3x_1 - 5x_2 &= 0 \end{aligned}\right\} \text{Augmented form}$$

x_3, x_4, x_5 - slack variables and the resultant equation is known augmented equations. (new variables are augmented)

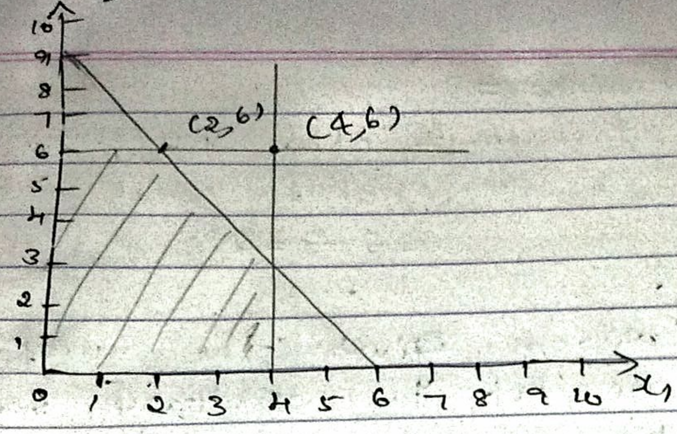
Let $x_1 = 3$, $x_2 = 2$ (arbitary values)

$x_3 = 1$, $x_4 = 8$ $x_5 = 5$

(3, 2, 1, 8, 5) - augmented solution

Augmented solution is a solution for original variables that has been augmented by the corresponding values of slag variables.

~~Refer~~ Refer the graph in page no. ⊛



Basic solution

take corner points (not feasible)

$x_1 = 4$ $x_2 = 6$

$x_3 = 0$ $x_4 = 0$ $x_5 = -6$

$(4, 6, 0, 0, -6) \longrightarrow$ basic solution.

Basic solution is ~~the~~ one of the corner point solution for x_1 & x_2.

Basic feasible solution

take corner points which is in feasible region

$x_1 = 2$ $x_2 = 6$

$x_3 = 2$ $x_4 = 0$ $x_5 = 0$

$(2, 6, 2, 0, 0) \longrightarrow$ basic feasible solution.

Basic feasible solution is the corner point solution in the feasible region.

Basic variables

The variables added are called basic variables (slack variables) (x_3, x_4, x_5)

Non basic variables = original variables (x_1, x_2)

total variables = 5 (BV + NBV)

No. of constraints = 3
Degree of freedom = total variables − no. of constraints
= 5 − 3 = 2

Degree of freedom gives the no. of variables which can be set to arbitary values to find the solution.
Degree of freedom = 2 means any 2 variables can be set to arbitary value (usually initial arbitary value is zero)
i.e $x_1 = 0$, $x_2 = 0$

Algebra of simplex method

Augmented form
(1) $x_1 + x_3 = 4$
(2) $2x_2 + x_4 = 12$
(3) $3x_1 + 2x_2 + x_5 = 18$
(0) $Z - 3x_1 - 5x_2 = 0$

Initial solution
$x_1 = 0$ $x_2 = 0$
$Z = 0$
(0, 0, 4, 12, 18) is the initial BFS

optimality test

Check if Z is optimal
it is not optimal because if we increase x_1 & x_2 Z increases.

Direction of movement

Increasing x_2, increases the z value more

Identify the leaving Basic variables

$2x_2 + x_4 = 12$

$x_4 = 12 - 2x_2 \Rightarrow x_2 \le 6$

$3x_1 + 2x_2 + x_5 = 18$

$x_1 = 0$

$x_5 = 18 - 2x_2$

$x_2 \le 9$

select the minimum

$\therefore x_2 \le 6$

assign $x_2 = 6$, $x_4 = 0$

$\therefore x_4$ is leaving, x_2 is entering

coefficients of leaving variable x_4

(equation)

$(0, 0, 1, 0)$

since x_2 takes place of x_4 - change coefficient of x_2 to x_4 coefficient.

new equation(2) = old equation(2) $\div 2$

new(2) = $x_2 + x_4/2 = 6$

new(1) = old(1) $\Rightarrow x_1 + x_3 = 4$

new(3) = old(3) − old(2)

$\Rightarrow 3x_1 - x_4 + x_5 = 6$

new(0) = old(0) + 5 × new(2)

$\Rightarrow z - 3x_1 + \frac{5}{2}x_4 = 30$ —— (0)

$x_1 + x_3 = 4$ —— (1)

$x_2 + x_4/2 = 6$ —— (2)

$3x_1 - x_4 + x_5 = 6$ —— (3)

Iteration 2:

optimality test

Increasing x_4, increase z $\therefore$ not optimal

x_4 - entering

Identify leaving

$x_1 + x_3 = 4$

$x_3 = 4 - x_1$ $\quad x_1 \leq 4$

$3x_1 - x_4 + x_5 = 6$

$x_5 = 6 - 3x_1$ $\quad x_1 \leq 2$

minimum $x_1 = 2$

x_1 - entering $\quad x_5$ - leaving

change coefficients of x_1 to x_5 coefficients

Coefficients of leaving variable x_5 (0,0,0,1)

~~cha~~ Chang

$Z + 3/2\, x_4 + x_5 = 36$

$Z + 3/2\, x_4 + x_5 = 36$ —— (0)

$x_3 + 1/3\, x_4 - 1/3\, x = 2$ —— (1)

$x_2 + 1/2\, x_4 = 6$ —— (2)

$x_1 - 1/3\, x_4 + 1/3\, x_5 = 2$ —— (3)

Solution is optimal, increasing x_4 or x_5

in Z equation will not increase Z

$Z = 36$, $x_1 = 2$, $x_2 = 6$

Tabular Form - Simplex

maximize $Z = 3x_1 + 5x_2$

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$, $x_1, x_2 \ge 0$

Augmented form

(0) $Z - 3x_1 - 5x_2 = 0$

(1) $x_1 + x_3 = 4$

(2) $2x_2 + x_4 = 12$

(3) $3x_1 + 2x_2 + x_5 = 18$

Iteration 0

BV	Eqn	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	-3	-5	0	0	0	0	
x_3	1	0	1	0	1	0	0	4	pivot row
x_4	2	0	0	2	0	1	0	12	6 min
x_5	3	0	3	2	0	0	1	18	9

pivot element pivot column

optimality Test

Solution is optimal if all the values in row 0 are positive.

In the example -3 and -5 are 2 -ve values in row 0. Therefore solution is not optimal.

Identify the most negative value in row 0. Mark that column. The column is pivot column and the variable is entering basic variable

x_1 is the entering BV

Divide RHS element by the respective pivot column elements which is greater than zero. Identify the minimum ratio. This test is known as minimum ratio test. The row with minimum ratio is pivot row and the variable is leaving basic variable

x_4 is leaving BV

The element which is common to pivot row & pivot column is pivot element.

2 is the pivot element.

Iteration 1:

Divide pivot row with pivot element and write the new row (row 2)

For the remaining rows make the coefficient of pivot column = 0 (Add or subtract from other rows by multiplying with constants if necessary)

B.V	Eq^n	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	$\boxed{-3}$	0	6	5/2	0	36	
x_1	1	0	1	0	1	0	0	4	=4
x_2	2	0	0	1	0	1/2	0	6	
x_5	3	0	3	0	0	-1	1	6	=

row(0) -3 -5 0 0 0 0

new row(2) 0 5 0 5/2 0 30

-3 0 0 5/2 0 30

new row3 = row(3) − old row(2)

3 2 0 0 1 18

0 2 0 1 0 12

3 0 0 −1 1 6

optimality test

Z row has −ve value

∴ solution is not optimal

repeat the process.

Iteration 2

BV	Equn	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	1	0	0	0	−3/2	1	36
x_3	1	0	0	0	1	1/3	−1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	−1/3	1/3	2

~~row~~ new(1) = old(1) − new(3)

1 0 1 0 0 4

1 0 0 −1/3 1/3 2

0 0 1 1/3 −1/3 2

new(0) = old(0) + 3 × new(3)

= −3 0 0 5/2 0 30

3 0 0 −1 1 6

0 0 0 −3/2 1 36

solution is optimal since all values in row(0) positive

$\therefore z = 36$

$x_1 = 8,\ x_2 = 6,\ x_3 = 2,\ x_4 =$ [illegible] $= x_5 = 0$

2) max $z = 3x_1 + 2x_2$

s.t $\quad x_1 + x_2 \le 4$

$\quad x_1 - x_2 \le 2$

Augmented form

(0) $\quad z - 3x_1 + \cdot 2x_2$

(1) $\quad x_1 + x_2 + x_3 = 4$

(2) $\quad x_1 - x_2 + x_4 = 2$

Iteration 0

BV	eqn	x_1	x_2	x_3	x_4	R.H.S	
z	0	$\boxed{-3}$	-2	0	0	0	= 0
x_3	1	1	1	1	0	4	: 4
x_4	2	1	-1	0	1	2	= 2 (min)

1 is the pivot element

x_1 is entering BV

x_4 is leaving BV

Iteration 1

BV	eqn	x_1	x_2	x_3	x_4	RHS
z	0	0	-5	0	3	6
x_3	1	0	2	1	-1	2
x_1	2	1	-1	0	1	2

row 1 > old(1) - old(2)

1 1 1 0 4

1 -1 0 1 2

---

0 2 1 -1 2

new row(0) = old(0) + 3 × old(2)

= −3	−2	0	0	0
3	−3	0	3	6
0	−5	0	3	6

2 pivot element

x_2 – entering BV

x_3 – leaving BV

Iteration 2

(divide pivot row with pivot element)

BV	eqn	x_1	x_2	x_3	x_4	RHS
Z	0	0	0	5/2	1/2	11
x_2	1	0	1	1/2	−1/2	1
x_1	2	1	0	1/2	−1/2	3

row0 = old(0) + 5 × new(1)   3−5/2

0	−5	0	3	6
0	5	5/2	−5/2	5
0	0	5/2	1/2	11

row 2 = old row2 + new row (1)

= 2	1	−1	0	1	2
1	0	1	1/2	−1/2	1
3	1	0	1/2	1/2	3

Since all values in row(0) is positive solution is optimal.

$\therefore Z = 11$

$x_1 = 3 \quad x_2 = 1 \quad x_3 = 0 \quad x_4 = 0$

3) max $z = 30x_1 + 25x_2$

s.t $2x_1 + x_2 \le 40$

$x_1 + 3x_2 \le 45$

$x_1 \le 12$

$x_1, x_2 \ge 0$

→ Augmented form

(0) $z - 30x_1 - 25x_2 = 0$

(1) $2x_1 + x_2 + x_3 = 40$

(2) $x_1 + 3x_2 + x_4 = 45$

(3) $x_1 + x_5 = 12$

Iteration 0

pivot column

BV	equn	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	-30	-25	0	0	0	0	
x_3	1	0	2	1	1	0	0	40	40/2 =
x_4	2	0	1	3	0	1	0	45	45/1 = 4
x_5	3	0	1	0	0	0	1	12	12/1 = 1

pivot element   pivot row

1 is the pivot element

x_1 is the entering B.V

x_5 is the leaving B.V.

Iteration 1

BV	equn	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	0	-25	0	0	30	360	
x_3	1	0	0	1	1	0	-2	16	16/1 =
x_4	2	0	0	3	0	1	-1	33	33/3 =
x_1	3	0	1	0	0	0	1	12	12/0 =

new(2) = old(2) − new(3)

1	3	0	1	0	45
1	0	0	0	1	12
0	3	0	1	−1	33

new(1) = old(1) − 2× new(3)

2	1	1	0	0	40
2	0	0	0	2	24
0	1	1	0	−2	16

new(0) = old(0) + 30× new(3)

−30	−25	0	0	0	0
30	0	0	0	30	360
0	−25	0	0	30	360

$40/2 = 20$

$45/1 = 45$

$12/1 = 1$ (min)

3 is the pivot element

x_2 is the entering B.V

x_4 is the leaving B.V

Iteration 2

BV	eqn	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	1	0	0	0	25/3	65/3	635
x_3	1	0	0	0	1	−1/3	−5/3	5
x_2	2	0	0	1	0	1/3	−1/3	11
x_1	3	0	1	0	0	0	1	12

new(2) = $\frac{\text{old}(2)}{3}$ = 0 1 0 1/3 −1/3 11

new(1) = old(1) − new(2)

0	1	1	0	−2	16
0	1	0	1/3	−1/3	11
0	0	1	−1/3	−5/3	5

$16/1 = 16$

$33/3 = 11$

$-2 + 1/3$

$\frac{-6+1}{3} = -5/3$

new(0) = old(0) + 25 × new(2)

		0	0	30	360
= 0	−25	0	25/3	−25/3	~~275~~
0	25	0	25/3	65/3	635
0	0	0			

all values in row(0) are +ve

$\therefore$ $Z = 635$

$x_1 = 12$ $\quad x_2 = 11$ $\quad x_3 = 5$ $\quad x_4 = x_5 = 0$

Tie breaking in Simplex

Case (1) : Tie for entering B.V

If there are two elements with the same value in 0th row of simplex table then tie for entering basic variable occurs. In such case we should go for arbitary selection. Answer will be same but no. of iteration may vary.

EX: Tie for entering B.V

$Z = 3x_1 + 3x_2$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

→ Augmented form

(0) $Z - 3x_1 + 3x_2 = 0$

(1) $x_1 + x_3 = 4$

(2) $2x_2 + x_4 = 12$

(3) $3x_1 + 2x_2 + x_5 = 18$

BV	equn	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	-3	-3	0	0	0	0	
x_3	1	0	1	0	1	0	0	4	$4/1=4$
x_4	2	0	0	2	2	1	0	12	$12/0=\infty$
x_5	3	0	3	2	0	0	1	18	$18/3=6$

Here we have 2 same minimum values in row(0)
select any one value arbitarily.

1 is the pivot element
x_1 is the entering BV
x_3 is the leaving BV

Iteration 1:

BV	equn	Z	x_1	x_2	x_3	x_4	x_5	RHS	
Z	0	1	0	-3	3	0	0	12	
x_1	1	0	1	0	1	0	0	4	$4/0=\infty$
x_4	2	0	0	2	0	1	0	12	$12/2=6$
x_5	3	0	0	2	-3	0	1	6	$6/2=3$

new (3) = old (3) − 3 × new (1)

3	2	0	0	1	18
3	0	3	0	0	12
0	2	-3	0	1	6

new (0) ~~new (2)~~ = old (0) + 3 × new (1)

-3	-3	0	0	0	0
3	0	3	0	0	12
0	-3	3	0	0	12

2 is the pivot element
x_2 is the entering BV

Iteration 2:

BV	equn	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	0	0	$-3/2$	0	$3/2$	21
x_1	1	1	0	1	0	0	4
x_4	2	0	0	3	1	-1	6
x_2	3	0	1	$-3/2$	0	$1/2$	3

$4/1 = 4$, $6/3 = 2$

$3 - 9/2$, $6 - \frac{9}{2}$

new(0) = old(0) + 3 x new(3)

$$\begin{array}{cccccc} 0 & -3 & 3 & 0 & 0 & 12 \\ 0 & 3 & -9/2 & 0 & 3/2 & 9 \\ \hline 0 & 0 & -3/2 & 0 & 3/2 & 21 \end{array}$$

new(2) = old(2) - 2 x new(3)

$$= \begin{array}{cccccc} 0 & 2 & 0 & 1 & 0 & 12 \\ 0 & 2 & -3 & 0 & 1 & 6 \\ \hline 0 & 0 & 3 & 1 & -1 & 6 \end{array}$$

Iteration 3:

BV	equn	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	0	0	0	$1/2$	1	24
x_1	1	1	0	0	$-1/3$	$1/3$	2
x_3	2	0	0	1	$1/3$	$-1/3$	2
x_2	3	0	1	0	$1/2$	0	6

new(0) = old(0) + $3/2$ new(2)

$$\begin{array}{cccccc} 0 & 0 & -3/2 & 0 & 3/2 & 21 \\ 0 & 0 & 3/2 & 1/2 & -1/2 & 3 \\ \hline 0 & 0 & 0 & 1/2 & 1 & 24 \end{array}$$

new(1) = old(1) - new(2)

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1/3 & -1/3 & 2 \\ \hline 1 & 0 & 0 & -1/3 & 1/3 & 2 \end{array}$$

new(3) = old(3) + $3/2$ new(2)

$$\begin{array}{cccccc} 0 & 1 & -3/2 & 0 & 1/2 & 3 \\ 0 & 0 & 3/2 & 1/2 & -1/2 & 3 \\ \hline 0 & 1 & 0 & 1/2 & 0 & 6 \end{array}$$

Case 2:

Ex:

→

(0)

(1)

(2)

BV

Z

x_3

x_4

BV

Z

x_3

x_2

Case 2: Tie for leaving BV

Ex: max $z = 3x_1 + 9x_2$

s.t $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

→ Augmented form

(0) $z - 3x_1 - 9x_2 = 0$

(1) $x_1 + 4x_2 + x_3 = 8$

(2) $x_1 + 2x_2 + x_4 = 4$

BV	equn	z	x_1	x_2	x_3	x_4	RHS	
z	0	1	-3	-9	0	0	0	
x_3	1	0	1	4	1	0	8	$8/4 = 2$
x_4	2	0	1	2	0	1	4	$4/2 = 2$

2 is the pivot element

x_2 is the entering BV

x_4 is the leaving BV

BV	equn	z	x_1	x_2	x_3	x_4	RHS
z	0	1	3/2	0	0	9/2	18
x_3	1	0	-1	0	1	-2	0
x_2	2	0	1/2	1	0	1/2	2

new(2) = old(2)/2

new(1) = old(1) − 4 × new(2)

=	1	4	1	0	8
	2	4	0	2	8
	-1	0	1	-2	0

new(0) = old(0) + 9 × new(2)

=	-3	-9	0	0	0
	9/2	9	0	9/2	8
	3/2	0	0	9/2	18

$-3 + 9/2$

$\frac{-6+9}{2} =$

row(0) has all values +ve.

$\therefore$ solution is optional

$z = 18$

$x_1 = 0$ $x_2 = 2$ $x_3 = 0$ $x_4 = 0$

If there are 2 variables with the same minimum ratio then tie for leaving BV occurs. (In the example variables x_3 and x_4 have same minimum ratio $= 2$) In such cases we need to select arbitarily. The no. of iteration varies. When there is a tie b/w 2 variables, the variable which is not selected reaches the value 0. This situation is known as decegeneracy. and the problem is degenerate problem.

minimization problem

Ex: minimize $z = x_1 + 2x_2 - 4x_3$

s.t $x_1 + x_2 + 2x_3 \leq 9$

$x_1 + x_2 - x_3 \leq 2$

$-x_1 + 2x_2 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0.$

$\rightarrow$ change z equation to maximization form (change sign throughout

max $-z = -x_1 - 2x_2 + 4x_3$

let $-z = z'$

$z' = -x_1 - 2x_2 + 4x_3$

Augmented form

(0) $z' + x_1 + 2x_2 - 4x_3 = 0$

(1) $x_1 + x_2 + 2x_3 + x_4 = 9$

(2) $x_1 + x_2 - x_3 + x_5 = 2$

(3) $-x_1 + 2x_2 + x_3 + x_6 = 4$

BV	equn	z'	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z'	0	1	1	2	-4	0	0	0	0
x_4	1	0	1	1	2	1	0	0	9
x_5	2	0	1	1	-1	0	1	0	2
x_6	3	0	-1	2	1	0	0	1	4

1 is the pivot element

x_3 is the entering BV

x_6 is the leaving BV

Iteration 1:

BV	equn	z'	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z'	0	1	-3	10	0	0	0	4	16
x_4	1	0	3	-3	0	1	0	-2	1 1/3
x_5	2	0	0	3	0	0	1	1	6
x_3	3	0	-1	2	1	0	0	1	4

new(0) = old(0) + 4 × new(3)

1	2	-4	0	0	0	0
-4	8	4	0	0	4	16
[illegible]	10	0	0	0	4	16

new(1) = old(1) − 2 × new(3)

1	1	2	1	0	0	9
-2	4	2	0	0	2	8
3	-3	0	1	0	-2	1

$new(2) = old(2) + new(3)$

1	1	-1	0	1	0	2
-1	2	1	0	0	1	4
0	3	0	0	1	1	6

3 is the pivot element

x_1 is entering BV

x_4 is leaving BV

BV	equn	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z'	0	1	0	7	0	1	0	2	17
x_1	1	0	1	-1	0	1/3	0	-2/3	1/3
x_5	2	0	0	3	0	0	1	1	6
x_3	3	0	0	1	1	1/3	0	1/3	13/3

$new(0) = old(0) + 3 \times new(1)$

-3	10	0	0	0	4	16
3	-3	0	1	0	-2	1
0	7	0	1	0	2	17

$new(3) = old(3) + new(1)$

-1	2	1	0	0	1	4	$1-\frac{2}{3}$
1	-1	0	1/3	0	-2/3	1/3	$4+\frac{1}{3}$
0	1	1	1/3	0	1/3	13/3	

row (0) contains all values +ve

$\therefore$ solution is optimal.

$z = 17$ i.e $z' = -z = -17$

$x_1 = 1/3$ $x_2 = x_4 = 0 = x_6$ $x_5 = 6$ $x_3 = 13/3$

//